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SOME OBSERVATIONS ON THE DYNAMIC BEHAVIOR OF COMPOSITES

Robert C. Reuter, Jr. and Stephen W. Tsai

September 1968

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FOREWORD

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ABSTRACT

An exploratory investigation of mathematical models for the description of wave propagation through unidirectional fiber-reinforced composites is reported. A self-consistent model using three concentric cylinders with different materials is studied. A pure shear wave propagation is presented. As another scheme of solving the dynamic behavior, fiber and matrix properties may be simulated by periodic functions, from which the elasticity theory of heterogeneous media can be applied.

Introduction

Dispersion in an elastic, non-dissipative, wave-supporting medium is a phenomenon resulting entirely from the interaction of stress waves with the boundaries of the medium. Single material waveguides admit wave reflections at their free boundaries, whereas stratified or composite waveguides give rise not only to reflections at their free boundaries but also to reflections and refractions of waves at interface boundaries. Mathematical representation of dispersion is accomplished through the so-called frequency equation which relates the phase velocity of the wave to its wavelength. When there is no dispersion, the phase velocity is constant and does not change with wavelength. This occurs, for example, in unbounded elastic homogeneous media where waves travel with the dilatation velocity $\left(C_D = [(\frac{\lambda + 2\mu}{\rho})^{\frac{1}{2}}]\right)$ and/or the shear wave velocity $(C_S = (\mu/\rho)^{\frac{1}{2}})$. Unbounded elastic composite media, however, cannot support waves without dispersion because of their multiple interface boundaries. Hence, an anisotropic, homogeneous representation of an elastic composite medium is insufficient for predicting its response to the propagation of stress waves. This illustrates a fundamental difference between the static and dynamic problem for composites.

Presented here are two possible modes for attempting to account for the dispersive nature of composire waveguides. A "self-consistent" model of a circular cylindrical composite waveguide with transverse isotropy is proposed. Three

concentric circular co-axial cylinders constitute the model. All three layers are elastic, isotropic and homogeneous. The properties of the outer shell are the "equivalent properties" of the composite waveguide being modeled. The core of the model has the properties of the encased fibers of the composite rod and the intermediate layer has the properties of the matrix material of the composite rod. The core and intermediate layer are included in the model so as to maintain the identity of the constituents of the prototype and their dimensions can be adjusted to control the dispersion of the composite rod. Another possibility for dispersive control is to exchange the properties of the core and intermediate layer of the model.

A second mode of dispersive representation is presented, based on the premise that material properties which can be expressed as a function of position give rise to dispersion. The potential of such a scheme is illustrated by a simple example. Material property functions for a square and hexagonal array of fiber encasements are proposed.

I. Self-Consistent Model

The concept of a "self-consistent" model has been introduced and a development of its frequency equation follows. Conditions under which a considerably simplified yet exact solution exists are demonstrated. Phase velocities which are equal to the conventional shear wave velocities of the constituent materials can exist under certain conditions imposed by the boundary conditions.

Three-layer stratification of the model is necessary so that enough geometric parameters are available to adjust the equivalent properties of the composite rod and its dispersion, independently.

In what follows, material properties, physical properties and configuration parameters are subscripted with a j, indicating reference to the core when $j = 1$, to the intermediate layer when $j = 2$, and to the outer layer when $j = 3$.

1. Solution for the Displacements

The equations of motion of three-dimensional, linear elasticity referred to cylindrical coordinates are

$$
\begin{aligned}
\rho_j \frac{\partial^2 u_j}{\partial t^2} &= (\lambda_j + 2\mu_j) \frac{\partial \Delta_j}{\partial r} - \frac{2\mu_j}{r} \frac{\partial \bar{\omega}_{zj}}{\partial \theta} + 2\mu_j \frac{\partial \bar{\omega}_{\theta j}}{\partial z} \\
\rho_j \frac{\partial^2 v_j}{\partial t^2} &= (\lambda_j + 2\mu_j) \frac{1}{r} \frac{\partial \Delta_j}{\partial \theta} - 2\mu_j \frac{\partial \bar{\omega}_{rj}}{\partial z} + 2\mu_j \frac{\partial \bar{\omega}_{zj}}{\partial r} \qquad (1) \\
\rho_j \frac{\partial^2 w_j}{\partial t^2} &= (\lambda_j + 2\mu_j) \frac{\partial \Delta_j}{\partial z} - \frac{2\mu_j}{r} \frac{\partial}{\partial r} (r\bar{\omega}_{\theta j}) + \frac{2\mu_j}{r} \frac{\partial \bar{\omega}_{rj}}{\partial \theta}
\end{aligned}
$$

where u_j, v_j and w_j are displacements in the r, θ and z directions, respectively, and

$$\Lambda_j = \frac{1}{r}\frac{\partial}{\partial r}(r u_j) + \frac{1}{r}\frac{\partial v_j}{\partial \theta} + \frac{\partial w_j}{\partial z}$$

$$2\bar{\omega}_{rj} = \frac{1}{r}\frac{\partial w_j}{\partial \theta} - \frac{\partial v_j}{\partial z}$$

$$2\bar{\omega}_{\theta j} = \frac{\partial u_j}{\partial z} - \frac{\partial w_j}{\partial r} \qquad (2)$$

$$2\bar{\omega}_{zj} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_j) - \frac{\partial u_j}{\partial \theta}\right]$$

The torsion problem for circular cylindrical rods is characterized by the following assumed solution.

$$u_j = w_j = 0$$

and $$v_j(r, z, t) = V_j(r) \exp\left[i(\gamma z + pt)\right] \qquad (3)$$

where $\gamma = 2\pi/\text{wavelength}$ and $p = 2\pi$ (frequency). Note that v_j is independent of θ. Solution (3) in conjunction with Equations (1) and (2) leads to a single ordinary differential equation in $V_j(r)$.

$$\frac{d^2V_j(r)}{dr^2} + \frac{1}{r}\frac{dV_j(r)}{dr} + \left[k_j - \frac{1}{r^2}\right] V_j(r) = 0 \qquad (4)$$

where $k_j = \left[\gamma^2\left(\frac{\rho_j p^2}{\mu_j \gamma^2} - 1\right)\right]^{1/2}$

Equation (4) is a Bessel equation of order 1 and has the following solution:

$$V_i = A_i J_1 (k_i r) + B_i Y_1 (k_i r) \tag{5}$$

Solutions for each layer of the stratified model follow immediately.

$$V_1 = A_1 J_1 (k_1 r) \tag{6}$$

$$V_2 = A_2 J_1 (k_2 r) + B_2 Y_1 (k_2 r) \tag{7}$$

$$V_3 = A_3 J_1 (k_3 r) + B_3 Y_1 (k_3 r) \tag{8}$$

In order for the displacements along the axis of the model to remain finite, $B_1 = 0$.

The k_i 's become

$$
\begin{aligned}
k_1 &= \left[\gamma^2 \left(\frac{\rho_f p^2}{\mu_f \gamma^2} - 1 \right) \right]^{1/2} \\
k_2 &= \left[\gamma^2 \left(\frac{\rho_m p^2}{\mu_m \gamma^2} - 1 \right) \right]^{1/2} \\
k_3 &= \left[\gamma^2 \left(\frac{\bar{\rho} p^2}{\bar{\mu} \gamma^2} - 1 \right) \right]^{1/2}
\end{aligned}
\tag{9}
$$

In this model the core material represents the encased fiber material of the prototype and the subscripts f and m refer to the fiber and matrix material, respectively. Bars over the properties in k_3 refer to the "equivalent properties" of the prototype.

2. Frequency Equation

Through the application of certain boundary conditions along the lateral boundaries of the model, the frequency equation is obtained. These boundary conditions are:

at $r = a$,

$$V_1 = V_2 \quad \text{and} \quad \sigma_{r\theta 1} = \sigma_{r\theta 2} \tag{10}$$

at $r = b$,

$$V_2 = V_3 \quad \text{and} \quad \sigma_{r\theta 2} = \sigma_{r\theta 3} \tag{11}$$

at $r = c$,

$$\sigma_{r\theta 3} = 0 \tag{12}$$

The form of solution (3) renders all other stress components zero. Application of the boundary conditions yields the following set of homogeneous equations in the constants A_i and B_i.

$$A_1 J_1(k_1 a) - A_2 J_1(k_2 a) - B_2 Y_1(k_2 a) = 0$$

$$\frac{\mu_f}{\mu_m} A_1 [k_1 a J_0(k_1 a) - 2J_1(k_1 a)] - A_2 [k_2 a J_0(k_2 a) - 2J_1(k_2 a)]$$

$$-B_2 [k_2 a Y_0(k_2 a) - 2Y_1(k_2 a)] = 0$$

$$A_2 J_1(k_2 b) + B_2 Y_1(k_2 b) - A_3 J_1(k_3 b) - B_3 Y_1(k_3 b) = 0 \tag{13}$$

$$\frac{\mu_m}{\mu} \left\{ A_2 [k_2 b J_0(k_2 b) - 2J_1(k_2 b)] + B_2 [k_2 b Y_0(k_2 b) - 2Y_1(k_2 b)] \right\}$$

$$-A_3 [k_3 b J_0(k_3 b) - 2J_1(k_3 b)] - B_3 [k_3 b Y_0(k_3 b) - 2Y_1(k_3 b)] = 0$$

$$A_3 [k_3 c J_0(k_3 c) - 2J_1(k_3 c)] + B_3 [k_3 c Y_0(k_3 c) - 2Y_1(k_3 c)] = 0$$

In order for a non-trivial solution for the A_i's and B_i's to exist, the determinant of their coefficients must vanish. Expansion of this singular determinant yields the frequency equation for torsional waves travelling in the self-consistent model. For each value of γ an infinite number of values of p will result, the first value of which is that corresponding to the first mode of propagation. Dispersion curves, sensitive to changes in material and physical parameters, are thus generated.

As previously mentioned, the self-consistent model contains sufficient parameters to adjust equivalent properties and dispersion independently. For the torsion problem, the equivalent properties will depend on the properties of the constituent materials of the prototype and its fiber volume fraction, ξ. Since the fiber volume fraction must be maintained throughout the self-consistent model, this will impose a restriction on the ratio, b/a ; that is, the ratio of the outer radius of the intermediate layer of the model to the radius of the core will depend on the fiber volume fraction, ξ. However, with the properties of the outer layer of the model depending on the fiber volume fraction, ξ, and not on the interface radii a, b or c, the ratio c/b is free to be chosen. Variation of the c/b ratio allows control over the dispersion of the self-consistent model without altering its equivalent properties.

3. Torsional Mode with No Dispersion

Under the conditions that both constituent materials have the same shear wave velocity (i.e., $\frac{\mu_f}{\rho_f} = \frac{\mu_m}{\rho_m} = \frac{\bar{\mu}}{\bar{\rho}}$) a considerably simplified solution results. Equations (4) reduce to one equation with its solution describing the displacements anywhere in the self-consistent model.

$$\frac{d^2V(r)}{dr^2} + \frac{1}{r}\frac{dV(r)}{dr} + [k - \frac{1}{r^2}]\, V(r) = 0 \tag{14}$$

where $k = \left[\gamma^2\left(\frac{\rho p^2}{\mu\gamma^2} - 1\right)\right]^{1/2}$

and where $\frac{\mu}{\rho} = \frac{\mu_f}{\rho_f} = \frac{\mu_m}{\rho_m} = \frac{\bar{\mu}}{\bar{\rho}}$

Equation (14) has a solution of the form

$$V(r) = AJ_1(kr) \tag{15}$$

subject only to the condition that $\sigma_{r\theta} = 0$ at the free boundary of the model. This condition is

$$kcJ_0(kc) - 2J_1(kc) = 0 \tag{16}$$

The first solution of this frequency equation is $k = 0$. Therefore, the phase velocity, which is given by p/γ, is equal to $(\mu/\rho)^{1/2}$. Equation (15) does not represent a solution for $k = 0$ and hence Equation (14) must be re-solved. Therefore, the new solution of Equation (14) with $k = 0$ is

$$V(r) = Ar + Br^{-1} \tag{17}$$

Since stresses and displacements must remain finite at the origin of r

$$V(r) = Ar \tag{18}$$

It is interesting to note that Equation (18) represents a solution for the first mode of propagation for the torsion problem of any composite waveguide provided that the

free boundary is circular and that all constituents have the same shear wave velocity. This is true for any number, shape and eccentricity of the encasements. Note further that this mode exists without dispersion; that is, the phase velocity is constant.

4. Special Solutions

Suppose now that the phase velocity of the self-consistent model is equal to either the shear wave velocity of the core or the shear wave velocity of the intermediate layer, with $\frac{\mu_f}{\rho_f} \neq \frac{\mu_m}{\rho_m}$. Since the following development is conceptually the same for either case, it will be assumed that the phase velocity is equal to the shear wave velocity of the core ($C_p = (\frac{\mu_f}{\rho_f})^{1/2}$). Under this circumstance, Equations (4) yield the following solutions for the self-consistent model:

$$
\begin{aligned}
V_1 &= A_1 r \\
V_2 &= A_2 J_1(k_2' r) + B_2 Y_1(k_2' r) \\
V_3 &= A_3 J_1(k_3' r) + B_3 Y_1(k_3' r)
\end{aligned}
\tag{19}
$$

where $$k_2' = \left[\gamma^2 \left(\frac{\rho_m \mu_f}{\rho_f \mu_m} - 1\right)^{1/2}\right]^{1/2}$$

and $$k_3' = \left[\gamma^2 \left(\frac{\bar{\rho} \mu_f}{\rho_f \bar{\mu}} - 1\right)\right]^{1/2}$$

Application of the boundary conditions (10), (11) and (12) yields the following set of homogeneous equations in the A_i's and B_i's.

$$aA_1 - J_1(k_2'a)A_2 - Y_1(k_2'a)B_2 = 0$$

$$[k_2'aJ_0(k_2'a) - 2J_1(k_2'a)]A_2 + [k_2'aY_0(k_2'a) - 2Y_1(k_2'a)] = 0$$

$$J_1(k_2'b)A_2 + Y_1(k_2'b)B_2 - J_1(k_3'b)A_3 - Y_1(k_3'b)B_3 = 0$$

$$\frac{\mu_m}{\mu}\left\{[k_2'bJ_0(k_2'b) - 2J_1(k_2'b)]A_2 + [k_2'bY_0(k_2'b) - 2Y_1(k_2'b)]B_2\right\} \tag{20}$$

$$- [k_3'bJ_0(k_3'b) - 2J_1(k_3'b)]A_3 - [k_3'bY_0(k_3'b) - 2Y_1(k_3'b)]B_3 = 0$$

$$[k_3'cJ_0(k_3'c) - 2J_1(k_3'c)]A_3 + [k_3'cY_0(k_3'c) - 2Y_1(k_3'c)]B_3 = 0$$

Again, for a non-trivial solution of the A_i's and B_i's to exist, the determinant of their coefficients must vanish. This singular determinant imposes a restriction on the values of γ (or wavelength) for which this mode can exist.

For waves travelling at the velocity of shear waves in the intermediate layer $\left(C_p = \left(\frac{\mu_m}{\rho_m}\right)^{1/2}\right)$, a similar treatment will yield an equation which imposes a restriction on the values of wavelength for which this mode can exist. Therefore, with the exception of the conditions of Art. (3), where $\frac{\mu_f}{\rho_f} = \frac{\mu_m}{\rho_m}$, waves travelling with the conventional shear wave velocity of either constituent material will be allowed only

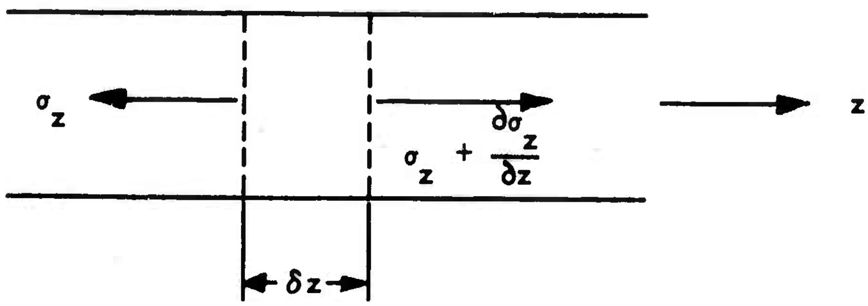
discrete values of wavelength. That is, unless $\frac{\mu_f}{\rho_f} = \frac{\mu_m}{\rho_m}$ the first mode of torsional wave propagation in the self-consistent model will not exist without dispersion.

II. Material Properties Which Vary with Position in the Waveguide

Another possible method of predicting the response of an elastic composite waveguide to the propagation of stress waves is that of expressing its material properties as functions of position. This method has the advantage of allowing the equations of motion for a homogeneous waveguide to be used, while the property variance accounts for the dispersive nature in the constitution of the waveguide. For example, the theory for homogeneous rods predicts that the first torsional mode is propagated with a phase velocity equal to $(\frac{\mu}{\rho})^{1/2}$ and that this mode exists without dispersion. The theory for the self-consistent model predicts that, in general, the first torsional mode exists with dispersion. A similar result is expected for the theory for homogeneous waveguides with material properties which vary with position.

1. One-Dimensional Wave Equation

In order to demonstrate the potential of the material property variance scheme consider the following simplified example. Let it be required to find the response of plane compressional waves. Let the waveguide have properties which somehow change with distance in the direction of its axis.



For simplicity, suppose that ρ (mass density) is constant and that E (modulus of elasticity) is equal to $E(z)$. Then, the equation of motion for an element in the waveguide is

$$\rho \delta z A \frac{\partial^2 u(z, t)}{\partial t^2} = \delta z A \frac{\partial \sigma_z(z, t)}{\partial z} \tag{1}$$

where A is the cross-sectional area.

For the one-dimensional problem, Hooke's Law is

$$\sigma_z(z) = E(z) \frac{\partial u(z)}{\partial z}$$

Equation (1) then becomes

$$\rho \frac{\partial^2 u(z, t)}{\partial t^2} = \frac{dE(z)}{dz} \frac{\partial u(z, t)}{\partial z} + E(z) \frac{\partial^2 u(z, t)}{\partial z^2}$$

Let a prime denote differentiation with respect to z. Then

$$\rho \frac{\partial^2 u(z, t)}{\partial t^2} = E'(z)\, u'(z, t) + E(z)\, u''(z, t) \tag{2}$$

Assume a solution in the form

$$u(z, t) = U(z) \cos(\omega t) \tag{3}$$

Then Equation (2) reduces to

$$\frac{d^2U(z)}{dz^2} + \frac{E'(z)}{E(z)}\frac{dU(z)}{dz} + \frac{\omega^2\rho}{E(z)}U(z) = 0 \tag{4}$$

When the wavelength is large compared to the lateral dimensions of the waveguide, an approximate solution can be obtained by assuming that

$$U(z) = A\cos(\gamma z) \tag{5}$$

where $\gamma = 2\pi/\text{wavelength}$.

Equation (5) is a solution of Equation (4) provided that the phase velocity is given by

$$C_p^2 = \frac{E'(z)\Lambda}{2\pi\rho}\tan\left(\frac{2\pi z}{\Lambda}\right) + \frac{E(z)}{\rho} \tag{6}$$

Therefore, material property variance does lead to dispersion in the one-dimensional wave problem, since the phase velocity depends on wavelength. If the modulus, $E(z)$, is constant, Equation (6) reduces to the conventional, one-dimensional phase velocity for compressional waves, $C_p = (E/\rho)^{1/2}$.

Although the above solution could lead to erroneous results (infinite phase velocities for certain values of z --this would depend on the form of $E(z)$), it does demonstrate the fact that material property variance can lead to dispersion in problems which do not predict dispersion when the properties are constant.

2. Variance Function for Square Array Fiber Packing

From this point on, only material property variance functions are proposed and no wave equations are solved. The square array for encased fiber packing lends

itself to a relatively simple property variance function and will be treated first. The fiber-reinforced waveguide which inspires this analysis is taken to be a cylindrical rod with its fibers parallel to the longitudinal axis of the rod. In the interest of simplicity, the following development is based on the x, y, z orthogonal coordinate system with z co-axial with the axis of the waveguide. If $\lambda(x, y)$ is any property of the waveguide which varies with position in the cross-section, the following property variance function is assumed for the square array.

$$\lambda(x, y) = \frac{A}{2}\left[\cos\left(\frac{2\pi x}{a}\right) + \cos\left(\frac{2\pi y}{a}\right)\right] + B \tag{7}$$

$\lambda(x, y)$ is a continuous function so that for any point in the cross-section both constituent materials contribute to the property in a meaningful way. Fiber spacing is the same in the x and y directions (for simplicity) and the property function has a period equal to this spacing in both directions. The constants A and B can be determined by enforcing certain predetermined behavior of $\lambda(x, y)$. These conditions are

a. $\lambda(ma, na) = \lambda_f$ (property of fiber) at the center of each fiber. Therefore

$$A + B = \lambda_f \tag{8}$$

b. $\lambda(ma, \frac{na}{2}) = \lambda(\frac{ma}{2}, na) = \bar{\lambda}$ at points intermediate between two adjacent fibers, where $\bar{\lambda}$ is the conventional equivalent property (for example, the law of mixtures). Therefore

$$B = \bar{\lambda} \tag{9}$$

Equation (9) can also be obtained in the following manner. Consider the equation

$$\frac{1}{A} \iint_A \lambda(x, y)\, dA = \bar{\lambda} \tag{10}$$

This equation states that the integral or $\lambda(x, y)$ over the cross-section of the waveguide, divided by its cross-sectional area is equal to the conventional equivalent property of the composite configuration. After inserting Equation (7) into Equation (10) and integrating cver the limits $x, y = -\beta$ to $+\beta$, the following result is obtained.

$$\bar{\lambda} = \frac{\alpha A}{2\pi\beta} \left[\sin \left(\frac{2\pi\beta}{\alpha} \right) \right] + B \tag{11}$$

If the dimensions of the waveguide are an integral number of fiber spacings, then Equation (11) reduces to

$$\bar{\lambda} = B$$

If not, however the fiber spacing is small compared to the lateral dimensions of the waveguide, then $\bar{\lambda} = B$. Figure 2 shows the square array and the performance of $\lambda(x, y)$ for one cycle along x and y (a), and also along $x = y$ (b), for the hypothetical case when $\lambda_f = 10$ psi, $\lambda_m = 5$ psi, $\xi = 0.8$ and $\bar{\lambda} = \lambda_f \xi + \lambda_m (1 - \xi)$.

When the law of mixtures is used as the equivalent property, there is a reasonable restriction imposed on the fiber fraction, ξ. At points equidistant from four fiber inclusions, $\lambda(ma/2, na/2) \geq \lambda_m$. This restricts the fiber fraction to $\xi \geq 1/2$. If $\xi = 1/2$, $\lambda(ma/2, na/2) = \lambda_m$.

3. Variance Function for Hexagonal Array Fiber Packing

A more complicated property variance function for hexagonal packing results,

as might be expected. Again, using λ as a general property which varies in the cross-section of the waveguide, the proposed form for $\lambda(r, \theta)$ is

$$\lambda(r, \theta) = A \cos \left\{\frac{\pi r}{3a} [3 + \sqrt{3}) + (3 - \sqrt{3}) \cos 6\theta]\right\}$$
$$+ \frac{B}{2} [3 - \cos 6\theta] \left[\cos \left\{\frac{2\pi r}{3a} [(3 + \sqrt{3}) + (3 - \sqrt{3}) \cos 6\theta]\right\} - 1\right] + C \tag{12}$$

The complication of this function requires some explanation. In order that $\lambda(r, \theta)$ reduce to the property of the encased fibers at the center of each, the function must have a period which varies with θ ; hence the form cos (cos). For $\theta = 0$, $\pi/3$, $2\pi/3$, . . . , the period is equal to the fiber spacing, a. At a/2 (intermediate between two adjacent fibers), $\lambda(r, \theta) = \bar{\lambda}$, where $\bar{\lambda}$ is the conventional equivalent property for the composite. In what follows, $\bar{\lambda}$ will be taken to be the law of mixtures. For $\theta = \pi/6$, $3\pi/6$, $5\pi/6$, . . . , $\lambda(r, \theta)$ has a period equal to $\sqrt{3}\,a$. Along this radius, $\lambda(r, \theta)$ must dip to a minimum at $r = \frac{\sqrt{3}\,a}{3}$ (points equidistant from three surrounding fibers), reach a secondary maximum at $r = \frac{\sqrt{3}\,a}{2}$ (points equidistant between two adjacent fibers), dip to another minimum at $r = \frac{2\sqrt{3}\,a}{3}$ and complete its cycle with a maximum of $\lambda(r, \theta) = \lambda_f$ at $r = \sqrt{3}\,a$. This behavior is illustrated in Figure 4 for the same hypothetical case used for the square array. The sum of two cos functions must be used so that the secondary maximum of $\lambda(r, \theta)$ can be obtained. This represents one of the conditions for the determination of the constants A, B and C. This condition can be separated into two parts.

a. In order for the secondary maximum to appear

$$A < 4B \tag{13}$$

b. The minimum values of $\lambda(r, \theta)$ will occur at $r = \frac{\sqrt{3}\,a}{3}$ and

$r = \frac{2\sqrt{3}\,a}{3}$ when

$$A = 2B \tag{14}$$

When $A = 4B$, no secondary maximum appears, however the minimum is relatively flat to accomodate fiber spacing large compared to the fiber radius, b. Conditions (13) and (14) lead to the coefficient, $\frac{B}{2}[3 - \cos 6\theta]$.

The two remaining conditions for the determination of A, B and C are

$$A + C = \lambda_f \tag{15}$$

and

$$-A + C = \bar{\lambda} \tag{16}$$

When $\bar{\lambda}$ is given by the law of mixtures, ($\bar{\lambda} = \lambda_f \xi + \lambda_m(1 - \xi)$), conditions (14), (15) and (16) lead to the following expressions for A, B and C.

$$A = \frac{4}{9}[\lambda_f(1 - \xi) + \lambda_m(\xi - 1)]$$

$$B = \frac{1}{18}[\lambda_f(1 - \xi) + \lambda_m(\xi - 1)] \tag{17}$$

$$C = \frac{1}{9}[\lambda_f(5 + 4\xi) - 4\lambda_m(\xi - 1)]$$

III. Conclusions

1) Dispersion is predicted for the first torsional mode by the general theory for the self-consistent model.

2) When the shear wave velocity of both constituents materials is the same $\left(\frac{\mu_f}{\rho_f} = \frac{\mu_m}{\rho_m}\right)$, the first torsional mode for the self-consistent model is propagated without dispersion.

3) Particular solutions, with discrete allowable values of wavelength can be obtained when the phase velocity is equal to the shear wave velocity of either constituent $\left(\text{when } \frac{\mu_f}{\rho_f} \neq \frac{\mu_m}{\rho_m}\right)$.

4) The self-consistent model appears plausible for predicting the response of composite waveguides with transverse isotropy in view of the fact that its equivalent properties and amount of dispersion can be adjusted independently.

5) Material property variance functions, when used in wave equations, represent a strong possibility for predicting the response of composite waveguides to the propagation of stress waves.

IV. Brief Outline of Immediate Future Work

1) Program frequency equation for self-consistent model and obtain dispersion curves for constant equivalent properties and various c/b ratios.

2) Develop a relation between c/b and the amount of dispersion in a prototype, composite waveguide.

3) Proceed to solve the wave equation with the square array property variance function. More sophisticated functions can be applied later.

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KEY WORDS

Wave Propagation

Dynamic Behavior

Fiber-reinforced Composites

Waveguides

Dispersion Composites

Self-consistent Model

Variance Function for Composites

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LIST OF FIGURES

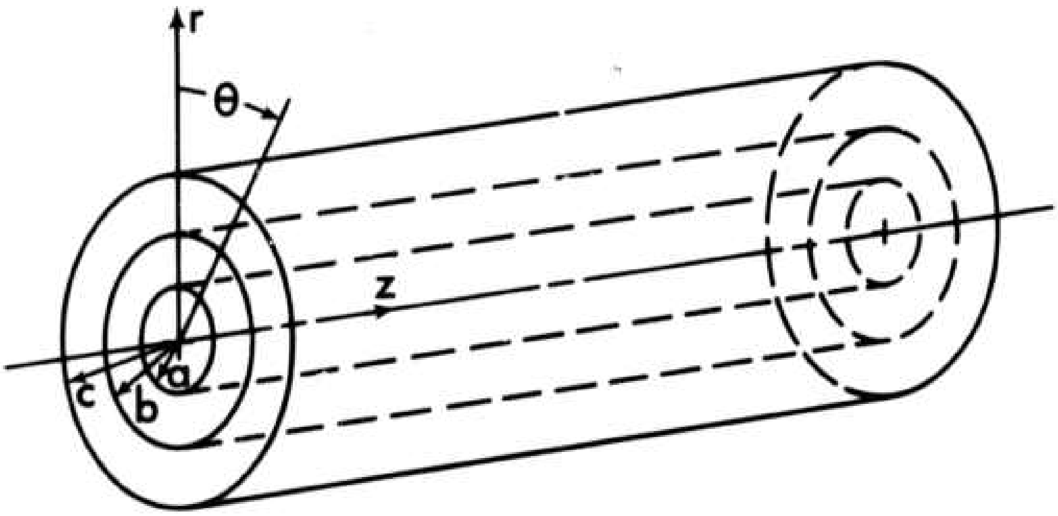

Fig. 1 Segment of Self-Consistent Model

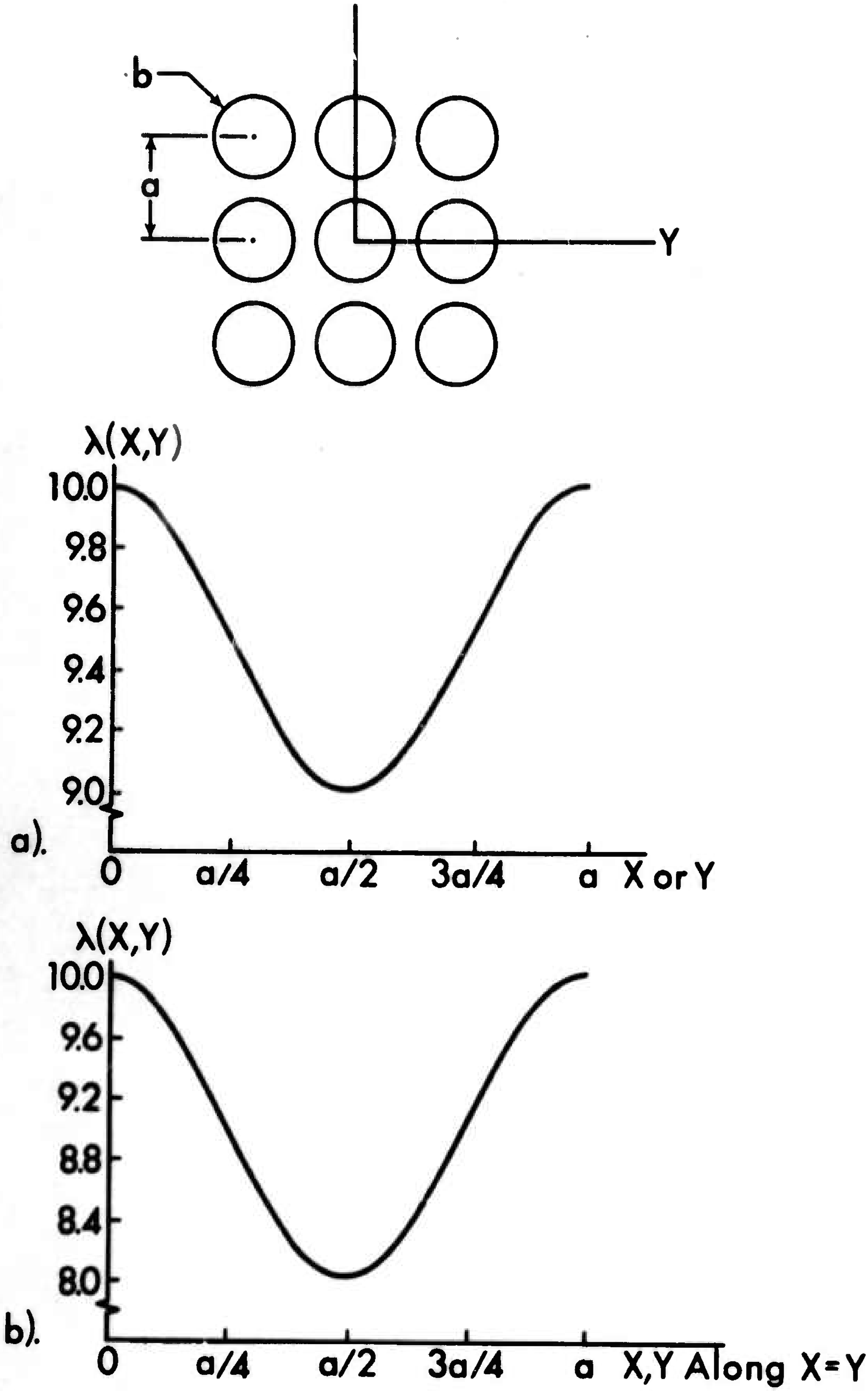

Fig. 2 Square Array Fiber Packing with Performance of Material Property Variance Function

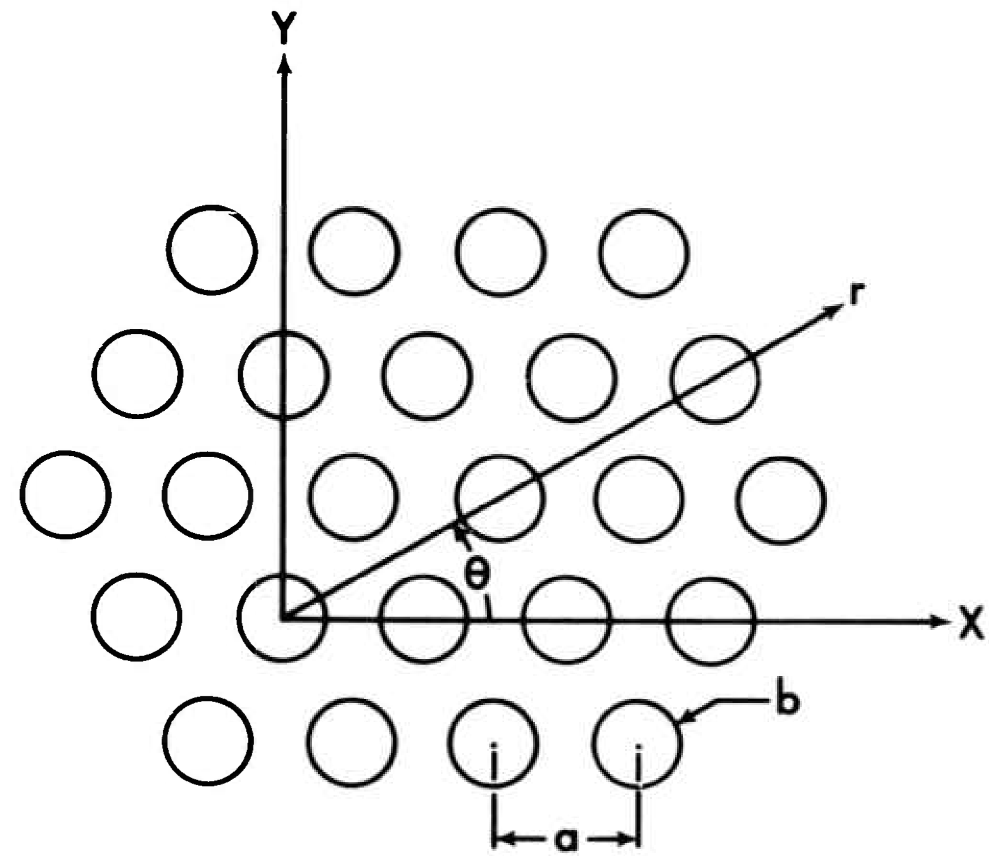

Fig. 3 Hexagonal Array of Fiber Packing

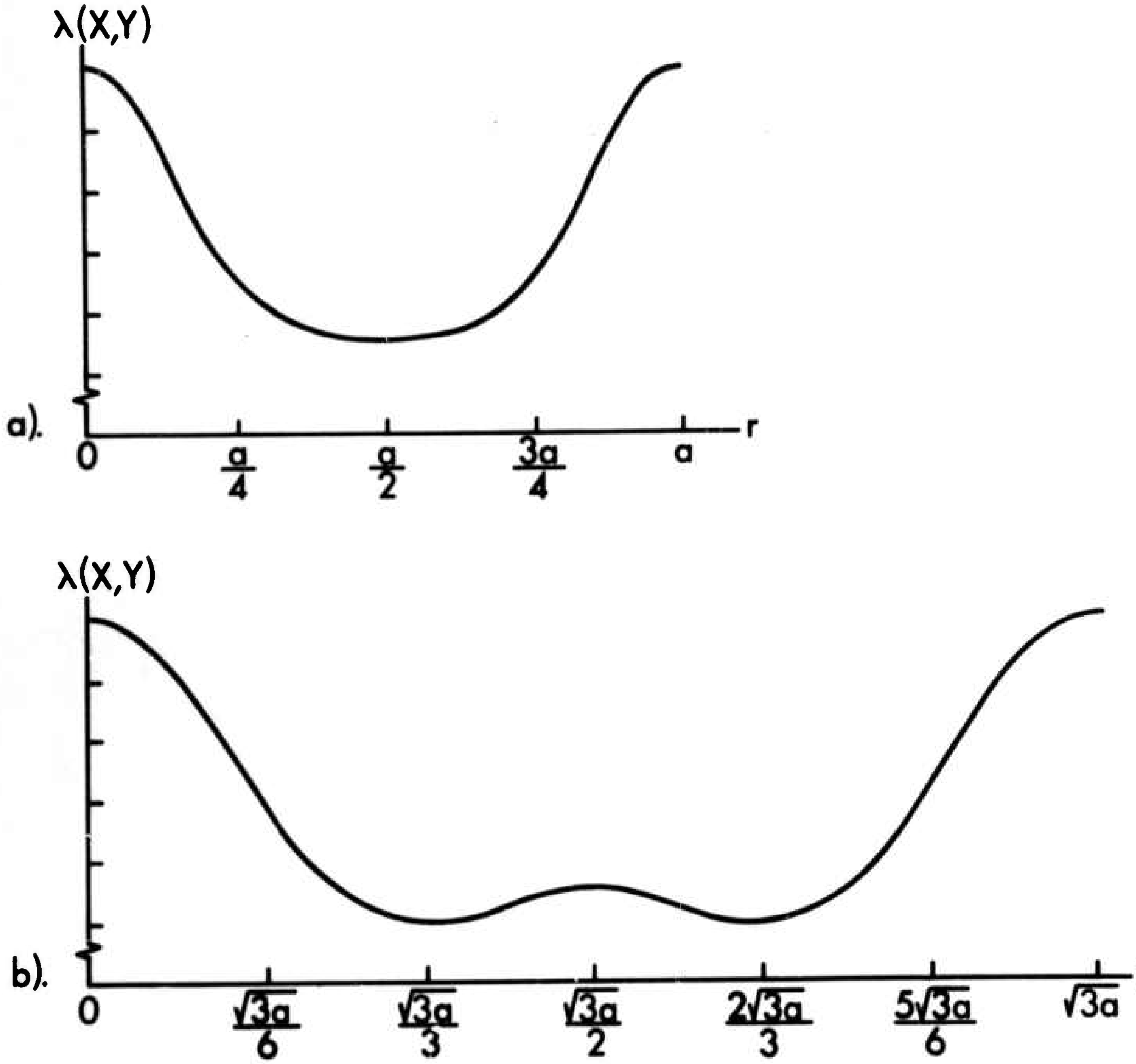

Fig. 4 Performance of Material Property Variance Function for Hexagonal Array. a). $\theta = 0, \pi/3, 2\pi/3, \ldots$; b). $\theta = \pi/6, 3\pi/6, 5\pi/6, \ldots$

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13. ABSTRACT

An exploratory investigation of mathematical models for the description of wave propagation through unidirectional fiber-reinforced composites is reported. A self-consistent model using three concentric cylinders with different materials is studied. A pure shear wave propagation is presented. As another scheme of solving the dynamic behavior, fiber and matrix properties may be simulated by periodic functions, from which the elasticity theory of heterogeneous media can be applied.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Wave Propagation						
Dynamic Behavior						
Fiber-reinforced Composites						
Waveguides						
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